Lecture 13: Beginning Function Spaces

- We have seen that separation of variables can generate families of product solutions for certain PDE, like for the heat equation. Superposition allows us to create finite linear combinations of these solutions to obtain more general solutions.

This motivates us to look at linear spaces, with additional limit structures in hopes of obtaining solutions by infinite series.

Inner Products & Norms

- Recall from 54 that $\mathbb{R}^n$ is a vector space equipped with an "inner product" $v \cdot w$. So $|v| = \sqrt{v \cdot v}$.

An inner product on a Complex vector space V is a function of two variables

$$u, v \in V \longmapsto \langle u, v \rangle \in \mathbb{C}$$

Satisfying

1.) Positive-Definite-ness: $\langle v, v \rangle \geq 0$ for $v \in V$, with equality iff $v = 0$

2.) Symmetry: for $v, w \in V$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$

3.) Linearity in the first variable: For $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2 \in V$

$$\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$$

$\rightsquigarrow$ the combination of 2.) & 3.) is called sesquilinearity.

- An inner product space is a real or complex vector space V equipped with an inner product $\langle \cdot, \cdot \rangle$

e.g.) $\mathbb{C}$ with $\langle v, w \rangle = v\bar{w}$

$C^0([0,1])$ with $\langle f, g \rangle = \int_0^1 f\bar{g}\,dx$

$M_n(\mathbb{C}) \rightsquigarrow$ Complex matrices with entries in $\mathbb{C}$ and $\langle A, B \rangle = tr(AB^*)$

Thm 7.1 Cauchy-Schwarz Inequality

For an inner product space V with $\|\cdot\|$ defined by

$\|u\| = \sqrt{\langle u,u\rangle}$ (the norm),

$$|\langle u,v\rangle| \le \|u\|\cdot\|v\|$$

for all $u, v \in V$.

Pf For $v, u \in V$ and $t \in \mathbb{R}$, consider

$$q(t) = \| u + t\langle u,v\rangle v\|^2$$

if $v=0$, the inequality is trivial. If $v \neq 0$,

$$q(t) = \langle u + t\langle u,v\rangle v,\ u + t\langle u,v\rangle v\rangle$$
$$= \|u\|^2 + 2t|\langle u,v\rangle|^2 + t^2|\langle u,v\rangle|^2\|v\|^2$$

which is a quadratic with minimum at $t_0 = -\|v\|^{-2}$:

$$q'(t) = 2|\langle u,v\rangle|^2 + 2t|\langle u,v\rangle|^2\|v\|^2 = 0 \quad \text{if}$$
$$t = -\|v\|^{-2}$$

Since $q(t) \ge 0$, $\quad 0 \le q(t_0) = \|u\|^2 - \dfrac{|\langle u,v\rangle|^2}{\|v\|^2}$ or

$$\|u\|^2\|v\|^2 \ge |\langle u,v\rangle|^2 \qquad \square$$

- The Cauchy-Schwarz Inequality shows that $\|\cdot\|$ is a norm by giving the triangle inequality: $\|u+v\| \le \|u\| + \|v\|$ as follows

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + 2\mathrm{Re}(\langle u,v\rangle) + \|v\|^2$$
$$\le \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$
$$\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$
$$\le (\|u\| + \|v\|)^2$$

- We may have norms not arising by inner products, such as for $f \in C^0(\bar{U})$ and $U \subseteq \mathbb{R}$ bounded

$$\sup\{|f(x)| : x \in U\} = \|f\|_\infty$$

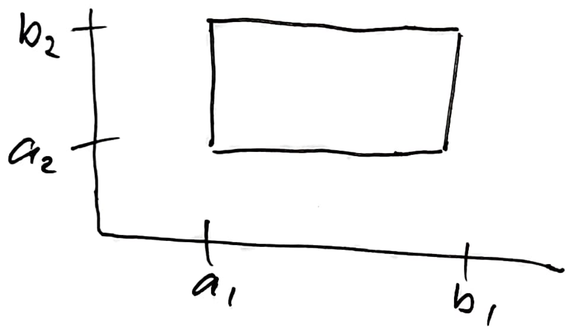
Basics of Lebesgue Integration

- Lebesgue integration was developed to accomodate more pathological functions than Riemann integration can, and also allows for more convergence properties. When a function is Riemann integrable, the two concepts agree, but Lebesgue integrals are more general.

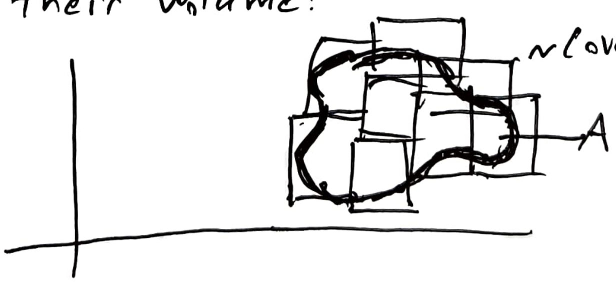
  → You can spend entire classes on Lebesgue Integration. We hit the <u>very basics</u>.

- The integral is based on the idea of generalizing a concept of volume or mass for sets in $\mathbb{R}^n$.

  For a rectangle $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = R \subseteq \mathbb{R}^n$, let $\text{Vol}(R) \stackrel{?}{=} (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$ as we normally do



- For a set $A \subseteq \mathbb{R}^n$, we approximate its volume by covering it with small rectangles & counting their volume:



Thus, we define the measure of A:

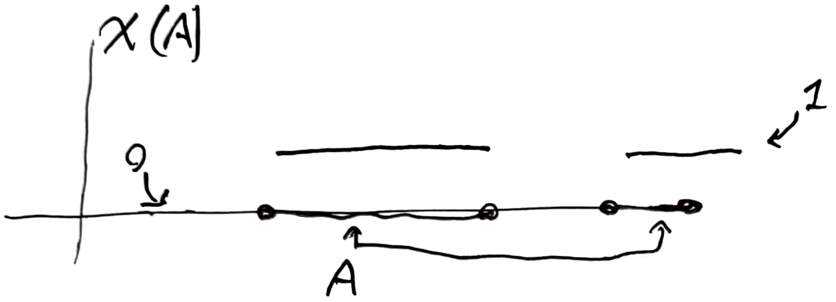
$$m(A) = \inf \{ \textstyle\sum_{j=1}^{\infty} \text{Vol}(R_j) ; \underbrace{A \subset \bigcup_{j=1}^{\infty} R_j}_{\text{Saying the rectangles cover A}} \}$$

& at-most countably many

- To give ourselves certain operations, we restrict to a class of sets which are measurable. This just rules out some pathological examples.

- We then build characteristic functions

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$



Then, $\underbrace{\int_{\mathbb{R}^n} \chi_A(x)\,dx}_{\text{"Should be the Volume of A"}}$ is defined to be $m(A)$.

We can then scale these and add them together:
for E_j disjoint,

$$\int_{\mathbb{R}^n} \sum_{i=1}^{m} \chi_{E_j}(x) \cdot c_j \, dx = \sum_{j=1}^{m} c_j \cdot m(E_j)$$

→ When functions are "nice enough" (measurable), we can approximate them by these sums to give integrals! We will assume all our functions are measurable.

→ If a set has measure 0; $m(A) = 0$, then $\int \chi_A \, dx = 0$. Thus, we usually ~~make~~ make an equivalence

$f \equiv g$ if $f = g$ except on a set of measure 0.

- Measure theory often has a concept of "almost everywhere" meaning that something occurs except on a set of measure 0. The above says $f \equiv g$ if $f = g$ almost everywhere.

$\boxed{\text{Lemma: 7.2}}$ For measurable $f, g : \Omega \to \mathbb{C}$ with $\Omega \subseteq \mathbb{R}^n$,

$$\int_\Omega |f-g| dx = 0 \quad \text{iff. } f \equiv g.$$

$\mathcal{L}^p$ Spaces

- A function is called integrable if its integral converges absolutely:

$$\int_\Omega |f| dx < \infty$$

For $p > 0$, we can define "p-integrable" functions

$$\mathcal{L}^p(\Omega) = \{ f : \Omega \to \mathbb{C} : \int_\Omega |f|^p dx < \infty \}$$

The case $0 < p < 1$ gives some weird properties, so we focus on $p \geq 1$.

- $\mathcal{L}^p(\Omega)$ is closed under scalar multiplication clearly: for $f \in \mathcal{L}^p$

$$\int |\alpha f|^p dx = |\alpha| \cdot \int |f|^p dx < \infty$$

Further, notice that $x \mapsto |x|^p$ is convex, for $p \geq 1$ so

$$\left| \frac{f+g}{2} \right|^p \leq \frac{|f|^p + |g|^p}{2}$$

giving that $\mathcal{L}^p(\Omega)$ is closed under addition

- $\mathcal{L}^p(\Omega)$ is a complex vector space!

$\|f\|_p = \left(\int_\Omega |f|^p dx \right)^{1/p}$ gives a norm.

• The case of $p=2$ also has an inner product!

$$\langle f, g \rangle = \int_\Omega f\bar{g}\,dx$$

• The $\mathcal{L}^p$-triangle inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$ is quite famous and called Minkowski's inequality. (it only holds $p \ge 1$). We will not prove this.

ex.) $h = a\chi_{[0,\ell]}$ for $a>0$, $\ell>0$

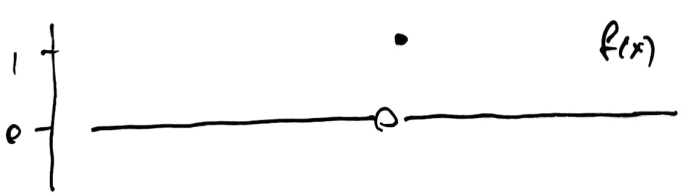
If $a=1$, $\|h\|_p = \left[\int_{\mathbb{R}^n} \chi_{[0,\ell]}^p\,dx\right]^{1/p} = \left[\int_{\mathbb{R}} \chi_{[0,\ell]}\,dx\right]^{1/p} = m([0,\ell])^{1/p} = \ell^{1/p}$

If $a \ne 1$, $\|h\|_p = a\ell^{1/p}$.

• The $\mathcal{L}^1$ norm gives us an idea of "total mass" $\|h\|_1 = a\ell$.
as p increases, we care more about a (height over spread)

$$\lim_{p\to\infty} \|h\|_p = a.$$

• This motivates a space $\mathcal{L}^\infty$ that just checks the "height" of a function. For continuous functions, the sup-norm $\|f\|_C = \sup\{|f(x)| : x \in \Omega\}$ is enough. For others, measure-0 issues interfere. Consider



$\leadsto$ Sup-norm is 1, but $f \equiv 0$ in integration.

· Thus, we define an essential supremum

$$\text{ess-sup}_{\mathbb{R}}(h) = \inf\{a \in \mathbb{R} : \{h > a\} \text{ has measure } 0\}$$

and $\|f\|_\infty = \text{ess-sup}(f)$,

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{C} : \|f\|_\infty < \infty\}$$

↳ these are thought of as the "bounded functions"

· We will mostly use $p = 1, 2$ or ∞.

ex.) Schrödinger Eqn. $\frac{\partial \psi}{\partial t} - i\Delta\psi = 0$.

You proved in the HW that

$$\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\psi(0,\cdot)\|_{L^2(\mathbb{R}^n)} \quad \text{(Energy!)}$$

Solutions also satisfy a "dispersion estimate"

$$\|\psi(t,\cdot)\|_\infty \leq \underbrace{C t^{-n/2}}_{} \|\psi(0,\cdot)\|_1$$

(By nonstationary phase). Decay over time or "loss" of amplitude

· Note: For Differentiability, a function f which is equivalent to a continuous/differentiable function is assumed to be represented by the continuous/differentiable function.